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EXISTENCE AND POSITIVITY RESULTS FOR THE $\varphi - \theta$ AND A MODIFIED $k - \varepsilon$ TWO-EQUATION TURBULENCE MODELS

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Abstract

In this work, we consider the $k - \varepsilon$ two-equation turbulence model from the point of view of mathematical and numerical analysis. We introduce a new equivalent two-equation model based on a change of variables. In the first part, we study this model with a constant eddy viscosity and where the coupling between the two-equation is done through the right hand side. Then we consider the complete model with a non linear viscosity. Finally, the impact of this new change of variables on the numerical computations with the $k - \varepsilon$ model is shown and a robust algorithm which preserves the positivity of the variables is given.

Keywords: Turbulence, two-equation models, existence, positivity.

RESULTATS D'EXISTENCE ET DE POSITIVITE POUR LE MODELE A DEUX EQUATIONS DE LA TURBULENCE $\varphi - \theta$ ET UN MODELE MODIFIE $k - \varepsilon$

Résumé

On s'intéresse au modèle $k - \varepsilon$ de turbulence du point de vue de l'analyse numérique et mathématique. Pour cela, on introduit un nouveau modèle équivalent basé sur un changement de variables. Dans une première partie, on étudie ce modèle avec une viscosité turbulente constante où le couplage entre les équations est assuré par les second membres. Ensuite on considère le modèle complet avec une viscosité non linéaire pour lequel on montre un résultat d'existence. Enfin, l'interêt numérique de ce nouveau changement de variables est souligné et un algorithme robuste qui préserve la positivité des variables est présenté.

Mots clés: Turbulence, modèles à deux équations, existence, positivité.

1 Introduction

Although the $k - \varepsilon$ two-equation turbulence model [LS1] is the most widely used turbulence model by engineers and scientists, from a mathematical point of view, the situation remains to be clarified. In [CA1], [MO1] the positivity of k and ε is studied by introducing a new variable $\theta = k/\varepsilon$.

In this work, we study the mathematical characteristics of the $\varphi-\theta$ model of turbulence. We begin by the equations of the problem (the Navier-Stokes and the $k-\varepsilon$ equations). Then, we describe how to obtain the $\varphi - \theta$ two-equation model. We study this model from the mathematical point of view by considering first an academic model problem where the eddy viscosity is assumed to be constant and then the general case. In particular, we give existence results for this model. All of these results can be extended to the $k-\varepsilon$ model if one supposes that the two models are equivalent (which is true in first order). Indeed, the differences come from the viscous terms. So, we are not able to treate the complete $k - \varepsilon$ model.

Finally, we describe the numerical implementation of the turbulence model and give a numerical example of turbulent flow computed by a mixed algorithm using the new change of variables during the transport step.

2 The Navier-Stokes equations and the $k-\varepsilon$ model

The eddy viscosity turbulence models are based on the introduction of a new viscosity for the fluid. In algebraic models, this quantity is related by simple algebraic expression to the mean flow. On the other hand, transport equations for turbulent scales can be introduced making it possible to evaluate the eddy viscosity. Dimensional analysis shows that two turbulent scales are needed to describe the eddy viscosity. This implies that we have to solve the classical Navier-Stokes equations and two transport diffusion equations.

Let ρ, u, E, p, T, e be the averaged density, velocity, total energy, pressure, temperature and internal energy of the fluid. We consider the following averaged and modelled Navier-Stokes equations:

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) &= 0 \\
\frac{\partial \rho u}{\partial t} + \nabla \cdot (\rho u \otimes u) &= \nabla \cdot \tau \\
\frac{\partial \rho E}{\partial t} + \nabla \cdot (\rho E u) &= \nabla \cdot (\tau u) + \nabla \cdot q
\end{aligned} \tag{2.1}$$

with

$$p = (\gamma - 1)\rho e, E = e + \frac{1}{2}u^2 + k, e = C_v T$$

$$\tau = -(p + \frac{2}{3}\rho k)I + (\mu + \mu_t)\{(\nabla u + \nabla u^T) - \frac{2}{3}(\nabla \cdot u)I\}, q = (K + K_t)\nabla e$$

$$K = \mu\frac{\gamma}{Pr}, K_t = \mu_t\frac{\gamma}{Pr_t}, \gamma = C_p/C_v$$

where k is the turbulent kinetic energy and μ_t the eddy viscosity (See [VA1] for the details of the averaging and the modelling).

The most popular two equations turbulence model is the $k - \varepsilon$ model [LS1]. This consists of two transport-diffusion equations dedicated to the description of two turbulent length scales (the turbulent kinetic energy k and the turbulent dissipation ε). From these turbulent scales, the eddy viscosity is computed by

$$\mu_t = c_\mu \rho \frac{k^2}{\varepsilon}$$

We denote by ρ the density of the fluid, u its velocity, $\nabla u = u_{i,j}$ the gradient of u, $D = u_{i,i}$ its divergence and $F = (\nabla u + \nabla u^t) : \nabla u - \frac{2}{3}D^2$. F is always positive. Indeed, if one supposes that the space dimension is two and if one denotes $u = (v, w)$, so

$$F = \frac{2}{3}(v_{,x} - w_{,y})^2 + (w_{,x} + v_{,y})^2 \geq 0.$$

The $k - \varepsilon$ equations are

$$\frac{\partial k}{\partial t} + u\nabla k - \frac{1}{\rho}\nabla((\mu + c_\mu \rho \frac{k^2}{\varepsilon})\nabla k) = S_k \tag{2.2}$$

$$\frac{\partial \varepsilon}{\partial t} + u\nabla \varepsilon - \frac{1}{\rho}\nabla((\mu + c_\varepsilon \rho \frac{k^2}{\varepsilon})\nabla \varepsilon) = S_\varepsilon \tag{2.3}$$

It can be seen that k and ε have different diffusion constants. The right hand sides of (2.2), (2.3) contain the production and the destruction terms for k and ε:

$$S_k = c_\mu \frac{k^2}{\varepsilon} F - \frac{2}{3}kD - \varepsilon \tag{2.4}$$

$$S_\varepsilon = c_1 k F - \frac{2}{3}\varepsilon D - c_2 \frac{\varepsilon^2}{k} \tag{2.5}$$

where $c_\mu, c_1, c_2, c_\varepsilon$ are respectively $0.09, 0.129, 1.92, 0.07$.

2.1 New $\varphi - \theta$ Turbulence Model; main result.

From a numerical point of view, there are two difficulties in using the $k - \varepsilon$ model. The positivity and the boundedness of k and ε are often violated by numerical schemes. Furthermore, it is well known that there may be an exponential blow up for k and ε if the viscous terms in (2.2),(2.3) are neglected. Several authors have proposed different changes of variables which make the system more stable. In [MO1], [MO2], [CA1] and [CA2] two new variables are proposed. The interest of this new system is that the right hand sides of the equations have the "suitable" sign, thereby avoiding the possibility of a blow up of the variables. More precisely, neglecting the viscous terms in the k and ε equations, we have tried to find a variable $\varphi = k^{\alpha}\varepsilon^{\beta}$ such that

$$\frac{d\varphi}{dt} = \frac{\partial\varphi}{\partial t} + u\nabla\varphi \leq 0.$$

In compressible situations ($D \neq 0$) only one pair

$$(\alpha,\beta) = (\frac{-c_1}{c_\mu}, 1)$$

makes φ dynamicaly stable. But, if $D = 0$ other choices are possible. For example,

$$(\alpha,\beta) = (-3, 2).$$

Moreover, in both cases, another variable (k/ε) appears in the equation of φ.
So, let

$$\theta = k/\varepsilon \quad \text{and} \quad \varphi = \varepsilon^2/\mathrm{k}^3. \tag{2.6}$$

In this work we study a new two-equation turbulent model based on θ and φ. The eddy viscosity becomes

$$\mu_t = c_\mu \rho \frac{1}{\theta\varphi} \tag{2.7}$$

and the θ and φ equations are obtained from the k and ε equations:

$$\frac{\partial\theta}{\partial t} + u\nabla\theta - Diff_\theta = -c_3 F\theta^2 + c_4 D\theta + c_5 \tag{2.8}$$

$$\frac{\partial\varphi}{\partial t} + u\nabla\varphi - Diff_\varphi = -c_6\theta\varphi F + c_7 D\varphi - c_8\frac{\varphi}{\theta} \tag{2.9}$$

where $Diff_\theta$ and $Diff_\varphi$ contain the terms provided by the viscous terms of the k and ε equations. We suppose that they are responsible for the diffusion of φ and θ. Therefore, they are modelled, introducing two new positive constants σ_φ and σ_θ, by:

$$Diff_\theta = \frac{1}{\rho}\nabla.((\mu + c_\theta\mu_t)\nabla\theta))$$

and

$$Diff_\varphi = \frac{1}{\rho}\nabla.((\mu + c_\varphi\mu_t)\nabla\varphi).$$

So, the $\varphi - \theta$ equations we consider are:

$$\frac{\partial \theta}{\partial t} + u\nabla\theta - \frac{1}{\rho}\nabla((\mu + c_\theta \mu_t)\nabla\theta) = -c_3 F\theta^2 + c_4 D\theta + c_5 \tag{2.10}$$

$$\frac{\partial \varphi}{\partial t} + u\nabla\varphi - \frac{1}{\rho}\nabla((\mu + c_\varphi \mu_t)\nabla\varphi) = -c_6\theta\varphi F + c_7 D\varphi - c_8\frac{\varphi}{\theta} \tag{2.11}$$

where the numerical constants in the RHS are obtained from the $k - \varepsilon$ equations. We have

$$c_3 = c_1 - c_\mu = 0.34, c_4 = \frac{2}{3}(\frac{c_1}{c_\mu} - 1) = 0.29, c_5 = c_2 - 1 = 0.92$$

$$c_6 = 3c_\mu - 2c_1 = 0.01, c_7 = 2 - \frac{2c_1}{3c_\mu} = 1.05, c_8 = 2c_2 - 3 = 0.84.$$

The diffusion constants c_φ and c_θ have to be numerically tuned. All of these constants are positive.

The initial and boundary conditions for θ and φ are

$$\theta(x,0) = \theta_0(x) > a \quad \varphi(x,0) = \varphi_0(x) > 0$$

and

$$\theta_{|\partial\Omega} = a > 0 \quad \varphi_{|\partial\Omega} = b > 0 \tag{2.12}$$

The assumption that the boundary and initial values are strictly positive comes from the physical modelling and the numerical techniques. It is well known that k vanishes at a solid wall and ε has a positive value. But the previous turbulence models are only valid for high Reynolds number regions (i.e. away from the wall). The most widely used approach to avoid this difficulty near the wall is to locate the boundary of the computational domain a small distance δ away from the wall and to give two positive Dirichlet conditions for k and ε. These conditions, called wall laws, are obtained from physical considerations [COU1]. Hence, on the computational boundary we have

$$k = \frac{u_\tau^2}{\sqrt{c_\mu}}, \quad \varepsilon = \frac{u_\tau^3}{\kappa\delta} \tag{2.13}$$

u_τ is given by the nonlinear expression

$$u_\tau(\frac{1}{0.4}log(\frac{\rho\delta u_\tau}{\mu}) + 5) = \vec{u}.\vec{n}$$

From (2.13), the boundary conditions on φ and θ are

$$\varphi = \frac{c_\mu^{3/2}}{.16\delta^2} = b, \quad \theta = \frac{.4\delta}{u_\tau\sqrt{c_\mu}} = a$$

In this work, we study this new model from the point of view of numerical and mathematical analysis. Our main result is:

Theorem 1:

Assume that the vector field u is such that:

$$u \in L^\infty([0,T);L^\infty(\Omega)), \quad F \in L^\infty([0,T);L^\infty(\Omega)), \quad D = 0 \textit{(divergence free hypothesis)}$$

and consider the problem (I):

$$\frac{\partial \theta}{\partial t} + u\nabla\theta - \nabla((\mu + c_\theta \mu_t)\nabla\theta) = -c_3 F\theta^2 + c_5$$

$$\frac{\partial \varphi}{\partial t} + u\nabla\varphi - \nabla((\mu + c_\varphi \mu_t)\nabla\varphi) = -c_6 F\theta\varphi - c_8\frac{\varphi}{\theta}$$

$$\varphi(t=0) = \varphi_0, \quad \theta(t=0) = \theta_0,$$

$$\varphi_{\partial\Omega} = b, \quad \theta_{\partial\Omega} = a$$

with

$$\theta_0 \in L^\infty(\Omega) \cap H^1(\Omega), \quad \varphi_0 \in L^\infty(\Omega),$$

$$\theta_0 \geq a, \quad 0 < \zeta \leq \varphi_0 \leq b \quad \textit{and} \quad 1 - a^2 > 0.$$

Then, for all $\mu \geq 0$ problem (I) has a solution $(\varphi - \theta)$ such that:

$$(i) \quad (\theta,\varphi) \in [(L^\infty([0,T), L^\infty(\Omega))) \cap (L^2([0,T), H^1(\Omega)))]^2,$$

$$(ii) \quad \inf_{[0,T)\times\Omega} \theta \geq a,$$

$$(iii) \quad \exists\lambda > 0 \quad \textit{such that} \quad \zeta e^{-\lambda T} \leq \varphi \leq b.$$

□

Remark

The free divergence hypothesis for u is not essential, and is made here for reasons of simplicity. The result is clearly true as soon as we suppose

$$\nabla.u \in L^2([0,T);L^\infty(\Omega)).$$

□

Firstly, we consider a model problem where the eddy viscosity is supposed to be constant. This means that the system is coupled only through the right hand side of the second equation. Thus, we may study the θ equation with zero Dirichlet boundary conditions, for which classical methods work. Next, we turn to the proof of theorem 1.

3 The model problem

In this section, the domain $\Omega \subset R^n$ for $n = 2, 3$ is supposed to be smooth, the vector field u is a known velocity field with sufficient regularity and we take the eddy viscosity $\mu + \mu_t$ constant and we denote by ν the corresponding kinematic viscosity. In other words we have $(\mu + \mu_t)/\rho = \nu$.
The φ, θ model is then

$$\frac{\partial \theta}{\partial t} + u\nabla\theta - \nu\Delta\theta = S_\theta \quad \text{in}[0,T) \times \Omega \tag{3.1}$$

$$\frac{\partial \varphi}{\partial t} + u\nabla\varphi - \nu\Delta\varphi = S_\varphi \quad \text{in}[0,T) \times \Omega \tag{3.2}$$

where S_φ and S_θ are as in (2.11) and (2.10) with Dirichlet boundary and strictly positive initial conditions for φ and θ. We notice that the first equation is completely autonomous. For this reason, we first focus our attention on equation (3.1). Even if the technique used to solve it is standard, we shall give some details on the estimates. Next, we make some remarks on equation (3.2) without going into detail. The zero Dirichlet boundary conditions are only chosen for reasons of simplicity. Thus, we consider the problem (I_M):

$$\begin{cases} \frac{\partial\theta}{\partial t} + u\nabla\theta - \nu\Delta\theta = S_\theta \\ \theta_{\partial\Omega} = 0 \\ \theta(t=0) = \theta_0 \geq 0 \end{cases}$$

3.1 Variational problem

We start with the variational formulation of our problem. In fact, we shall consider the following equation:

$$\frac{\partial \theta}{\partial t} + u\nabla\theta - \nu\Delta\theta = -c_3 F\theta\,|\theta| + c_4 D\theta + c_5 \tag{3.3}$$

As will be seen, a positive solution of (3.3) is also a solution of (3.1) as soon as $\theta_0 \geq 0$. Equation (3.3) must be understood in a distributional sense, that is (for example):

$$-\int_0^T dt \int_\Omega dx \frac{\partial\phi}{\partial t}\theta - \int_\Omega dx\theta\phi(0,x) - \int_0^T dt \int_\Omega dx\theta(\phi D + u\nabla\phi) + \nu\int_0^T dt\int_\Omega dx\nabla\theta\nabla\phi =$$
$$-c_3\int_0^T dt \int_\Omega dx F\theta\,|\theta|\,\phi + c_4\int_0^T dt\int_\Omega dx D\theta\phi + c_5\int_0^T dt\int_\Omega dx\phi \tag{3.4}$$

where $\phi \in C_c^\infty([0,T) \times \Omega)$. For instance, consider the convective term

$$\int_0^T dt \int_\Omega dx\theta(\phi D + u\nabla\phi) \leq$$

$$\|\phi\|_\infty \int_0^T dt \|D\|_\infty \int_\Omega dx\,|\theta| + \|\nabla\phi\|_\infty \int_0^T dt \int_\Omega dx\,|\theta u|\,.$$

To make sense of the convective and viscous term, we require:

$$\theta \in L^\infty([0,T);L^2(\Omega)), u \in L^2([0,T);L^\infty(\Omega)), D \in L^2([0,T);L^\infty(\Omega))$$

and

$$\nabla\theta \in L^2([0,T),L^2(\Omega)).$$

For the non linear term we have

$$\int_0^T dt \int_\Omega dx\,|\theta|\,\theta\phi F \leq \|\phi\|_\infty \int_0^T dt\|F\|_\infty \int_\Omega dx|\theta|^2.$$

So to make sense of the variational equation, we require at least

$$\theta \in L^\infty([0,T);H_0^1(\Omega))$$

$$u \in L^2([0,T);L^\infty(\Omega)), F \in L^2([0,T);L^\infty(\Omega)), D \in L^2([0,T);L^\infty(\Omega)). \tag{3.5}$$

□

The main result of this section is:

Theorem 3.1

Assume

$$u \in L^\infty([0,T);L^\infty(\Omega)), \quad D \in L^2([0,T);L^\infty(\Omega)), \quad F \in L^2([0,T);L^\infty(\Omega)),$$

$$\theta_0 \in L^\infty(\Omega)\cap H_0^1(\Omega), \quad \theta_0 \geq 0,$$

then problem (I_M) has a unique solution θ such that:

$$\theta \in L^\infty([0,T),L^\infty(\Omega)) \cap L^2([0,T);H_0^1(\Omega))$$

and

$$\theta \geq 0 \quad in \quad \Omega\times[0,T).$$

Proof (sketch):

Thanks to the a priori estimates for θ in $L^2([0,T),H_0^1(\Omega))$ and $\theta_{,t}$ in $L^2([0,T);H^{-1}(\Omega))$ given by lemma 3.1, the classical Faedo-Galerkin method [LI1] can be used to obtain the existence of a solution for the θ equation in $L^2([0,T);H_0^1(\Omega))$. The positivity and the uniqueness of the solution are proved respectively in lemma 3.2 and 3.3. □

3.2 Lemma 3.1, A priori estimates for θ

If $\theta_0 \in L^\infty(\Omega) \cap H_0^1(\Omega)$ and if we have u, F and D as in (3.5) then we have $\theta \in L^\infty([0,T); L^\infty(\Omega))$, $\nabla\theta \in L^2([0,T); L^2(\Omega))$ and $\theta_{,t} \in L^2([0,T); L^2(\Omega))$.

Proof:

We denote $\theta_{,t}$ by θ'. We consider equation (3.3) with zero Dirichlet boundary condition and positive initial condition $\theta = \theta_0$. Working then in $H_0^1(\Omega)$ we multiply (3.3) by θ and integrate:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\theta|^2 + \nu\int_\Omega |\nabla\theta|^2 = -\int_\Omega c_3 F|\theta|^3 + \int_\Omega c_4 D\theta^2 + \int_\Omega c_5\theta - \int_\Omega u.\nabla(\theta)\theta. \qquad (3.6)$$

Since F is always positive and due to Young's inequality ($|\theta| \leq \frac{1}{p}|\theta|^p + \frac{p-1}{p}$), we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\theta|^2 \leq c_4\int_\Omega D\theta^2 + C' + C\int_\Omega |\theta|^2 + \left|\int_\Omega u.\nabla(\theta)\theta\right|. \qquad (3.7)$$

with $C, C' > 0$. Integrating by parts the last term of the RHS we have

$$\int_\Omega u.\nabla(\theta)\theta = -\frac{1}{2}\int_\Omega D\theta^2.$$

Thus

$$\left|\int_\Omega u.\nabla(\theta)\theta\right| \leq \frac{1}{2}\|D\|_\infty \int_\Omega \theta^2$$

and inequality (3.7) then becomes

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\theta|^2 \leq C\int_\Omega \theta^2 + C'. \qquad (3.8)$$

So,

$$\int_\Omega |\theta|^2 \leq e^{2CT}\int_\Omega |\theta_0|^2 + C"$$

where $C = (c_4 + \frac{1}{2})\|D\|_\infty \in L^2([0,T))$, $C' = c_5|\Omega|^{1/2}$ and $C" = \frac{C'}{C}(e^{2CT} - 1)$ are positive. Therefore,

$$\theta \in L^\infty([0,T); L^2(\Omega))$$

In addition, by (3.6) and since F is always positive, we have

$$\nu\int_0^T\int_\Omega |\nabla\theta|^2 \leq c_4\int_0^T\int_\Omega D|\theta|^2 + c_5\int_0^T\int_\Omega \theta - \int_0^T\int_\Omega u.\nabla(\theta)\theta + \frac{1}{2}\int_\Omega |\theta_0|^2. \qquad (3.9)$$

By a similar argument we then have

$$\int_0^T\int_\Omega |\nabla\theta|^2 \leq C \qquad (3.10)$$

In other terms,

$$\nabla\theta \in L^2([0,T);L^2(\Omega)).$$

Therefore, θ stays in $L^2([0,T),H_0^1(\Omega))$. We notice that if in addition we have θ_0 in $L^p(\Omega)$ for $1 \le p < \infty$, we obtain in the same way θ in $L^\infty([0,T),L^p(\Omega))$. Indeed, the estimates are obtained multiplying the θ equation by $|\theta|^{p-2}\theta$, integrating and still using the fact that $F \ge 0$, which is crucial. The domain Ω being bounded, we deduce:

$$\theta \in L^\infty([0,T);L^\infty(\Omega)) \cap L^2([0,T);H_0^1(\Omega)).$$

It is easy to see that θ' is bounded in $L^2([0,T);H^{-1}(\Omega))$. In fact, we have a priori estimate for $\theta' = \frac{\partial\theta}{\partial t}$ in $L^2([0,T);L^2(\Omega))$. To do so, we multiply (formally) (3.1) by θ' and integrate in space and time:

$$\int_0^T\int_\Omega |\theta'|^2 + \int_0^T\int_\Omega u\nabla\theta\theta' + \nu\int_0^T\int_\Omega \nabla\theta\nabla\theta'$$

$$= -c_3\int_0^T\int_\Omega F\theta^2\theta' + c_4\int_0^T\int_\Omega D\theta\theta' + c_5\int_0^T\int_\Omega \theta' \tag{3.11}$$

At first, we notice that

$$\nu\int_0^T\int_\Omega \nabla\theta\nabla\theta' = \frac{\nu}{2}\int_0^T\int_\Omega (|\nabla\theta|^2)',$$

since $\nabla\theta$ is in $L^2([0,T);L^2(\Omega))$ and θ_0 in $L^\infty(\Omega)\cap H_0^1(\Omega)$,

$$\frac{\nu}{2}\int_0^T\int_\Omega (|\nabla\theta|^2)' = \frac{\nu}{2}\left(\int_\Omega |\nabla\theta(T,.)|^2 - \int_\Omega |\nabla\theta_0|^2\right).$$

The other expressions are clearly bounded. The bounds are obvious to obtain using the bounds on θ, F and D. Indeed, we have

$$\left|\int_0^T\int_\Omega u\nabla\theta\theta'\right| \le \|u\|_{L^\infty([0,T);L^\infty(\Omega))}\|\nabla\theta\|_{L^2([0,T);L^2(\Omega))}\|\theta'\|_{L^2([0,T);L^2(\Omega))},$$

$$\left|\int_0^T\int_\Omega F\theta^2\theta'\right| \le \|F\|_{L^2([0,T);L^\infty(\Omega))}\|\theta\|_{L^\infty([0,T);L^\infty(\Omega))}\|\theta'\|_{L^2([0,T);L^2(\Omega))},$$

and

$$\left|\int_0^T\int_\Omega D\theta\theta'\right| \le \|D\|_{L^2([0,T);L^\infty(\Omega))}\|\theta\|_{L^\infty([0,T);L^\infty(\Omega))}\|\theta'\|_{L^2([0,T);L^2(\Omega))}.$$

Therefore,

$$\|\theta'\|_{L^2([0,T);L^2(\Omega))} \le C\|\theta'(0,.)\|_{L^2(\Omega)}.$$

In other terms,

$$\theta' \in L^2([0,T);L^2(\Omega)) \tag{3.12}$$

□

3.3 A Maximum Principle for θ

For numerical and physical reasons it is important that θ remains positive if it is positive at time $t = 0$. In [MO1], [CA1] one can find a proof of this positivity in the absence of viscous terms using the method of characteristics. Here a maximum principle for θ is given. This maximum principle guarantees the positivity of θ even in the presence of the diffusion terms.

Lemma 3.2

For zero Dirichlet boundary condition and positive initial condition, θ remains positive.

Proof:

We write $\theta = \theta^+ - \theta^-$ with θ^+ and θ^- positive. We assume that $\theta_0^- = 0$. We have $\theta^-_{\partial\Omega} = 0$ because $\theta_{\partial\Omega} = 0$. We then multiply the θ equation by $-\theta^-$

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left|\theta^-\right|^2 + \nu\int_\Omega \left|\nabla\theta^-\right|^2 = -c_3\int_\Omega F(\theta^-)^3 + c_4\int_\Omega D(\theta^-)^2 - c_5\int_\Omega \theta^- - \int_\Omega \frac{1}{2}D(\theta^-)^2 \tag{3.13}$$

Therefore

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left|\theta^-\right|^2 \le (c_4 + \frac{1}{2})\|D\|_\infty \int_\Omega \left|\theta^-\right|^2 \tag{3.14}$$

Combined with the fact that $\theta_0^- = 0$, we obtain $\theta^- = 0$. □

3.4 Uniqueness of θ

lemma 3.3

Problem (I) has a unique solution θ in $L^\infty([0,T); H_0^1(\Omega))$ if F and D are in $L^2([0,T); L^\infty(\Omega))$.

Proof:

Let v_1 and v_2 be two solutions of (3.1) in $L^\infty([0,T), H_0^1(\Omega))$ for the same boundary and initial conditions and $w = v_1 - v_2$. So $w(0,x) = 0$ and $w(t, \partial\Omega) = 0$. The w equation is

$$\frac{\partial w}{\partial t} + u\nabla w - \nu\Delta w = -c_3 F w(v_1 + v_2) + c_4 D w \tag{3.15}$$

Multiplying (3.15) by w and integrating we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega w^2 - \int_\Omega \frac{w^2}{2}D + \nu\int_\Omega |\nabla w|^2 = -c_3\int_\Omega F w^2(v_1+v_2) + c_4\int_\Omega D w^2 \tag{3.16}$$

We take F and $D \in L^2([0,T); L^\infty(\Omega))$ and, due to the previous positivity result for θ, v_1 and v_2 are positive. So,

$$\frac{d}{dt}\int_\Omega w^2 \le \|D\|_\infty (c_4 + \frac{1}{2})\int_\Omega w^2$$

So,

$$\int_\Omega w^2 \leq (\int_\Omega w_0^2) \exp((c_4 + \frac{1}{2})\|D\|_\infty t)$$

Therefore $w = 0$ in Ω because $w_0 = 0$. □

Remark

Suppose that in the model problem we replace the homegenous Dirichlet boundary condition by $\theta = \theta_{\partial\Omega}$ on $\partial\Omega$, we obtain the same results as previously by putting $\tilde{\theta} = \theta - \theta_{\partial\Omega}$. □

3.5 A regularity result for θ

In the previous section, it is shown that (3.1) admits a solution in $L^\infty([0,T); L^\infty(\Omega)) \cap L^2([0,T); H_0^1(\Omega))$ if $\theta_0 \in L^\infty(\Omega)$. We shall prove that in fact the solution is more regular.

We consider the problem (I):

$$\frac{\partial \theta}{\partial t} + u\nabla\theta - \nu\Delta\theta = S_\theta$$

$$\theta_{\partial\Omega} = 0 \quad \text{and} \quad \theta(t=0) = \theta_0.$$

Proposition 3.1

Under the hypothesis of theorem 3.1, θ is in $L^\infty([0,T); L^\infty(\Omega) \cap H_0^1(\Omega)) \cap L^2([0,T); H^2(\Omega))$.

Proof:

We use the method of translation (differential quotient method), originally introduced by L.Nirenberg [Br1]. For reasons of simplicity, we extend θ and u by zero outside of Ω. We denote:

$$D_h\theta(x) = \frac{1}{|h|}[\theta(x+h) - \theta(x)],$$

Multiplying (I) by $D_{-h}D_h\theta$ and integrating by parts, we first remark that:

$$-\int_\Omega \Delta\theta D_{-h}D_h\theta = \int_\Omega |\nabla D_h\theta|^2 \tag{3.17}$$

and

$$\int_\Omega \frac{\partial\theta}{\partial t} D_{-h}D_h\theta = \frac{1}{2}\frac{d}{dt}\int_\Omega |D_h\theta|^2. \tag{3.18}$$

In the same way,

$$\left|\int_\Omega S_\theta D_{-h}D_h\theta\right| \leq \|S_\theta\|_2^{1/2} (\int_\Omega |D_{-h}D_h\theta|^2)^{1/2}. \tag{3.19}$$

A trivial consequence of the a priori estimates obtained in lemma 3.1 is $S_\theta \in L^\infty([0,T);L^2(\Omega))$. By using proposition $(XI.3)$ in [Br1]:

$$\int_\Omega |D_{-h}D_h\theta|^2 \leq \int_\Omega |\nabla D_h\theta|^2.$$

and Young's inequality:

$$\left(\int_\Omega |\nabla D_h\theta|^2\right)^{1/2} \leq \frac{1}{\alpha} + \alpha \int_\Omega |\nabla D_h\theta|^2,$$

(where $\alpha > 0$ will be specified below) we obtain:

$$\left|\int_\Omega S_\theta D_{-h}D_h\theta\right| \leq \|S_\theta\|_{L^\infty([0,T);L^2(\Omega))}\left(\frac{1}{\alpha} + \alpha\int_\Omega |\nabla D_h\theta|^2\right). \tag{3.20}$$

Now, we consider the term:

$$\left|\int_\Omega u\nabla\theta D_{-h}D_h\theta\right| \leq C\int_\Omega |\nabla\theta|\,|D_{-h}D_h\theta|,$$

where $C = \|u\|_{L^\infty([0,T);L^\infty(\Omega))}$. Still using Young's inequality,

$$|\nabla\theta|\,|D_{-h}D_h\theta| \leq \frac{1}{\alpha}|\nabla\theta|^2 + \alpha|D_hD_{-h}\theta|^2,$$

and by proposition $(XI.3)$ in [Br1] we obtain:

$$\left|\int_\Omega u\nabla\theta D_{-h}D_h\theta\right| \leq \frac{C}{\alpha}\int_\Omega |\nabla\theta|^2 + \alpha C\int_\Omega |\nabla D_h\theta|^2. \tag{3.21}$$

Combining (3.17), (3.18), (3.20) and (3.21) and choosing α such that

$$\delta = \alpha(C + \|S_\theta\|_{L^\infty([0,T);L^2(\Omega))}) < \nu$$

we obtain the estimate:

$$\frac{d}{dt}\int_\Omega |D_h\theta|^2 + (\nu-\delta)\int_\Omega |\nabla D_h\theta|^2 \leq \alpha C\int_\Omega |\nabla\theta|^2 + \frac{1}{\alpha}\|S_\theta\|_{L^\infty([0,T);L^2(\Omega))}, \tag{3.22}$$

from which we deduce that:

$$\int_\Omega |D_h\theta|^2 \leq \left(\alpha C\|\theta\|^2_{L^2([0,T);H^1_0(\Omega))} + \frac{1}{\alpha}\|S_\theta\|_{L^\infty([0,T);L^2(\Omega))}\right)T.$$

Using Fatou's lemma, we obtain:

$$\nabla\theta \in L^\infty([0,T);L^2(\Omega)).$$

So,

$$\theta \in L^\infty([0,T]; L^\infty(\Omega) \cap H_0^1(\Omega)).$$

Still from (3.22), we arrive at:

$$(\nu-\delta)\int_0^T\int_\Omega |\nabla D_h\theta|^2 \le (\alpha C\|\theta\|_{L^\infty([0,T];H_0^1(\Omega))} + \frac{1}{\alpha}\|S_\theta\|_{L^\infty([0,T];L^2(\Omega))})T + \int_0^T\int_\Omega |D_h\theta|^2,$$

but,

$$\int_0^T\int_\Omega |D_h\theta|^2 \le \int_0^T\int_\Omega |\nabla\theta|^2,$$

then

$$\theta \in L^2([0,T]; H^2(\Omega)).$$

Now, we notice that

$$\nabla S_\theta \in L^2([0,T), L^2(\Omega))$$

because we have $\theta \in L^\infty([0,T]; L^\infty(\Omega))$, $\nabla\theta \in L^\infty([0,T), L^2(\Omega))$, $F \in L^2([0,T]; L^\infty(\Omega))$ and $D \in L^2([0,T]; L^\infty(\Omega))$. We denote by ∂ any partial derivative (time and space). Let ϕ be in $C_c^\infty([0,T); \Omega)$; from (3.1), multiplying by $\partial\phi$ we have:

$$\int_Q \frac{\partial\theta}{\partial t}\partial\phi + \int_Q u.\nabla\theta\partial\phi + \nu\int_Q \nabla\theta\nabla(\partial\phi) = \int_Q S_\theta\partial\phi,$$

where $Q = [0,T) \times \Omega$. Consequently we have:

$$\int_Q \frac{\partial(\partial\theta)}{\partial t}\phi + \int_Q u\nabla(\partial\theta)\phi + \nu\int_Q \nabla(\partial\theta)\nabla\phi =$$

$$\int_Q \partial S_\theta\phi - \int_Q \partial u.\nabla\theta\phi$$

In the distribution sense, $\partial\theta$ then satisfies:

$$\frac{\partial(\partial\theta)}{\partial t} + u\nabla(\partial\theta) - \nu\Delta(\partial\theta) = \partial S_\theta - \partial u.\nabla\theta. \tag{3.23}$$

The previous work and equality (3.23) show how the regularity of θ is related to the regularity of u. Indeed, assume that

$$\partial u \in L^\infty([0,T); L^\infty(\Omega)),$$

then, using (3.5) and applying the same argument, we obtain:

$$\partial\theta \in L^\infty([0,T); H^1_{loc}(\Omega)) \quad \text{and} \quad \nabla(\partial\theta) \in L^2([0,T); H^1_{loc}(\Omega))$$

Therefore,

$$\nabla\theta \in L^\infty([0,T); L^\infty(\Omega)),$$

$$\nabla^2\theta \in L^\infty([0,T);L^2(\Omega)),$$
$$\theta \in L^2([0,T);H^3(\Omega)).$$

□

Remarks:

1. The previous regularity results are local in Ω because we have no information (a priori) on the derivative of θ at the boundary of Ω. In a same way, if we now suppose that

$$\partial^2 u \in L^\infty([0,T);L^\infty(\Omega)) \tag{3.24}$$

then

$$\partial^2\theta \in L^\infty([0,T),L^\infty(\Omega)) \quad \text{and} \quad \theta \in L^\infty([0,T);H^3(\Omega)).$$

By Sobolev's embedding theorem, the solution of (3.1) then satisfies:

$$\theta \in L^\infty([0,T),C^1_{loc}(\Omega))$$

□

2. Strong maximum principle

Assume that (3.24) holds and let θ in $L^\infty([0,T),C^1_{loc}(\Omega))$ be the solution of (3.1). Moreover, suppose that θ_0 strictly positive and that at a point $(t_0,x_0)\in]0,T[\times\Omega$ we have $\theta(x_0)=0=\min\theta$. We already know that θ is positive for all (x,t) then we have at this point (using proposition 3.1):

$$\frac{\partial\theta}{\partial t}=\nabla\theta=0, \quad \Delta\theta\geq 0,$$

But from (3.1) we have

$$-\nu\Delta\theta(x_0,t_0)=c_5>0,$$

which is a contradiction. Consequently

$$\theta>0 \quad \text{in}[0,T[\times\Omega,$$

which is a strong maximum principle. □

3.6 Variational formulation of φ equation.

In the distribution sense, (3.2) is ($\phi \in C^\infty_c([0,T)\times\Omega)$)

$$-\int_0^T dt\int_\Omega dx\frac{\partial\phi}{\partial t}\varphi-\int_\Omega dx\varphi\phi(0,x)-\int_0^T dt\int_\Omega dx\varphi(\phi D+u\nabla\phi)+\nu\int_0^T dt\int_\Omega dx\nabla\varphi\nabla\phi=$$
$$-c_6\int_0^T dt\int_\Omega dxF\varphi\theta\phi+c_7\int_0^T dt\int_\Omega dxD\varphi\phi-c_8\int_0^T dt\int_\Omega dx\frac{\varphi}{\theta}\phi \tag{3.25}$$

To make sense (3.25), we require then at least

$$\varphi \in L^\infty([0,T),H^1(\Omega)), \theta \in L^\infty([0,T),H^1(\Omega)), \frac{1}{\theta} \in L^\infty([0,T),L^2_{loc}(\Omega))$$

$$u \in L^2([0,T);L^2(\Omega)), D \in L^2([0,T);L^\infty(\Omega)), F \in L^2([0,T);L^\infty(\Omega))$$

Notice that this makes our results local (away from the boundaries). But, because the term in $1/\theta$ has the suitable sign, it is possible to have global estimates for φ. Indeed, it is easy to see that

$$\int_\Omega |\varphi|^2 \leq \int_\Omega |\varphi_0|^2 exp((c_7+\frac{1}{2})\|D\|_\infty T) \tag{3.26}$$

$$\int_\Omega |\nabla\varphi|^2 \leq \frac{1}{\nu}\int_\Omega |\varphi_0|^2 exp((c_7+\frac{1}{2})\|D\|_\infty T). \tag{3.27}$$

Therefore, we have φ in $L^\infty([0,T);H^1_0(\Omega))$.

3.7 A maximum principle for φ.

Lemma 3.4

For zero Dirichlet boundary and positive initial conditions, φ remains postitive.

Proof:

Again, we write $\varphi = \varphi^+ - \varphi^-$. Multiplying (3.2) by $-\varphi^-$ and integrating we have $(\theta \geq 0)$

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left|\varphi^-\right|^2 + \nu\int_\Omega \left|\nabla\varphi^-\right|^2 \leq c_7\|D\|_\infty \int_\Omega \left|\varphi^-\right|^2. \tag{3.28}$$

Therefore,

$$\|\varphi^-\|^2 \leq \|\varphi_0^-\|^2 exp(\|D\|_\infty(c_7+\frac{1}{2})T) \tag{3.29}$$

So, $\varphi^- = 0$ at all times because $\varphi_0^- = 0$. □

3.8 Uniqueness of φ

For the uniqueness of φ we work as for θ. Indeed, let φ_1 and φ_2 be two solutions obtained with the same boundary and initial conditions. Let $w = \varphi_1 - \varphi_2$. It is easy to see that we have

$$\|w\|^2 \leq \|w_0\|^2 exp(\|D\|_\infty(c_7+\frac{1}{2})T).$$

So $w = 0$ at all times because $w_0 = 0$. □

4 The real system

Recalling the $\varphi - \theta$ equations, we consider:

$$\frac{\partial \theta}{\partial t} + u\nabla\theta - \nabla((\mu + \frac{1}{\theta\varphi})\nabla\theta) = -c_3 F\,|\theta|\,\theta + c_5 \tag{4.1}$$

$$\frac{\partial \varphi}{\partial t} + u\nabla\varphi - \nabla((\mu + \frac{1}{\theta\varphi})\nabla\varphi) = -c_6 F\theta\varphi - c_8\frac{\varphi}{\theta} \tag{4.2}$$

with

$$\zeta < \varphi(t=0) = \varphi_0 \leq b, \theta(t=0) = \theta_0 \geq a,$$

$$\varphi_{\partial\Omega} = b, \theta_{\partial\Omega} = a$$

with

$$0 < a \leq 1 \quad 0 < \zeta < b.$$

The goal of this section is to prove the following theorem:
Theorem 1.
Assume that the vector field u is such that:

$$u \in L^\infty([0,T); L^\infty(\Omega)), \quad F \in L^\infty([0,T); L^\infty(\Omega)), \quad D = 0 \textit{(divergence free hypothesis)}$$

and that θ_0 and φ_0 are in $L^\infty(\Omega)$. Then, for all $\mu \geq 0$ problem ((4.1)-(4.2)) has a solution $(\varphi - \theta)$ such that:

$$(i) \quad (\theta, \varphi) \in [(L^\infty([0,T), L^\infty(\Omega))) \cap (L^2([0,T), H^1(\Omega)))]^2,$$

$$(ii) \quad \inf_{[0,T)\times\Omega} \theta \geq a,$$

$$(iii) \quad \exists\lambda > 0 \quad \textit{such that} \quad \zeta e^{-\lambda T} \leq \varphi \leq b$$

$$\textit{with} \quad \lambda \geq |\Omega|\,(a\|F\|_{L^\infty(\Omega)} + \frac{1}{a}).$$

Proof:
We introduce the following perturbed system (I_ϵ):

$$\frac{\partial \theta_\epsilon}{\partial t} + u\nabla\theta_\epsilon - \nabla(H_\epsilon(\theta_\epsilon, \varphi_\epsilon)\nabla\theta_\epsilon) = -c_3 F\,|\theta_\epsilon|\,\theta_\epsilon + c_5 \tag{4.3}$$

$$\frac{\partial \varphi_\epsilon}{\partial t} + u\nabla\varphi_\epsilon - \nabla(H_\epsilon(\theta_\epsilon, \varphi_\epsilon)\nabla\varphi_\epsilon) = -\varphi_\epsilon(c_6\theta_\epsilon F + c_8 K_\epsilon(\theta_\epsilon)) \tag{4.4}$$

with

$$\theta_\epsilon(t=0) = \theta_0 \geq a, \quad \zeta < \varphi_\epsilon(t=0) = \varphi_0 \leq b$$

$$\theta_{\epsilon|\partial\Omega} = a, \quad \varphi_{\epsilon|\partial\Omega} = b$$

$$K_\epsilon(f) = \frac{1}{(\epsilon^2 + f^2)^{1/2}}$$

$$H_\epsilon(f,g) = \mu + \frac{1}{(\epsilon^2 + f^2 g^2)^{1/2}}$$

Thanks to lemma 4.2, we have a priori estimates for θ_ϵ and φ_ϵ in $L^\infty([0,T); L^\infty(\Omega)) \cap L^2([0,T); H^1(\Omega))$ and for θ'_ϵ and φ'_ϵ in $L^2([0,T); H^{-1}(\Omega))$. Moreover, it is easy to see that $\theta_\epsilon \geq a$. The proof is the same as in the constant eddy viscosity case.

Now, we can pass to the limit in (I_ϵ). Indeed, let $(\theta^n_\epsilon, \varphi^n_\epsilon)$ be a sequence of approximate solutions of (I_ϵ) (for instance obtained by the Faedo-Galerkin method). Thanks to the $L^\infty([0,T), L^\infty(\Omega)) \cap L^2([0,T); H^1(\Omega))$ bounds on $(\theta^n_\epsilon, \varphi^n_\epsilon)$ and the $L^2([0,T); H^{-1}(\Omega))$ bounds on $(\theta'_\epsilon, \varphi'_\epsilon)$, $(\theta^n_\epsilon, \varphi^n_\epsilon)$ is weakly compact in $(L^2([0,T); H^1(\Omega)))^2$ and strongly compact in $(L^q(Q))^2$,$(q \geq 2)$. Consequently, if we denote by θ_ϵ and φ_ϵ the limit (extracting subsequences, always denoted $(\theta^n_\epsilon, \varphi^n_\epsilon)$), then $(\theta^n_\epsilon, \varphi^n_\epsilon) \to (\theta_\epsilon, \varphi_\epsilon)$ a.e. in $[0,T) \times \Omega$.

We focus our attention on the behaviour of the term

$$\int_0^T \int_\Omega \nabla.(H_\epsilon(\theta^n_\epsilon, \varphi^n_\epsilon)\nabla\theta^n_\epsilon)\phi,$$

ϕ being any test function. The other terms will be treated in an obvious way. Indeed, the techniques of the constant eddy viscosity case remain valid for them. Thus we compute (after translation $\tilde{\theta}_\epsilon = \theta_\epsilon - a, \tilde{\varphi}_\epsilon = \varphi_\epsilon - b$):

$$\int_0^T \int_\Omega H_\epsilon(\tilde{\theta}_\epsilon{}^n, \tilde{\varphi}_\epsilon{}^n)\nabla\tilde{\theta}_\epsilon{}^n.\nabla\phi - H_\epsilon(\tilde{\theta}_\epsilon, \tilde{\varphi}_\epsilon)\nabla\tilde{\theta}_\epsilon.\nabla\phi$$

$$= \int_0^T \int_\Omega [H_\epsilon(\tilde{\theta}_\epsilon, \tilde{\varphi}_\epsilon)(\nabla\tilde{\theta}_\epsilon{}^n - \nabla\tilde{\theta}_\epsilon) + \nabla\tilde{\theta}_\epsilon{}^n(H_\epsilon(\tilde{\theta}_\epsilon{}^n, \tilde{\varphi}_\epsilon{}^n) - H_\epsilon(\tilde{\theta}_\epsilon, \tilde{\varphi}_\epsilon))]\nabla\phi.$$

From the fact that $\tilde{\theta}_\epsilon{}^n \to \tilde{\theta}_\epsilon$ weakly in $L^2([0,T); H^1_0(\Omega))$ and that

$$\left|H_\epsilon(\tilde{\theta}_\epsilon, \tilde{\varphi}_\epsilon)\right| \leq \frac{1}{\epsilon}$$

we deduce that

$$\int_0^T \int_\Omega H_\epsilon(\tilde{\theta}_\epsilon, \tilde{\varphi}_\epsilon)[\nabla\phi(\nabla\tilde{\theta}_\epsilon{}^n - \nabla\tilde{\theta}_\epsilon)] \to 0 \quad when \quad n \to \infty$$

In addition, using the finite increments formula, we have:

$$|H_\epsilon(x_1, y_1) - H_\epsilon(x_2, y_2)| \leq C_\epsilon \, |x_1 y_1 - x_2 y_2| \,, \tag{4.5}$$

where C_ϵ is a positive constant which only depends on ϵ. From (4.5) and combined with the fact that $\tilde{\theta}_\epsilon{}^n$ and $\tilde{\varphi}_\epsilon{}^n$ are weakly compact in $L^\infty([0,T);L^\infty(\Omega))$ and $\nabla\tilde{\theta}_\epsilon{}^n$ is bounded in $L^2([0,T);L^2(\Omega))$, we are able to deduce that

$$\int_0^T\int_\Omega \nabla\tilde{\theta}_\epsilon{}^n\nabla\phi[H_\epsilon(\tilde{\theta}_\epsilon{}^n,\tilde{\varphi}_\epsilon{}^n)-H_\epsilon(\tilde{\theta}_\epsilon,\tilde{\varphi}_\epsilon)]\to 0 \quad when \quad n\to\infty.$$

We have the same analysis on the term:

$$\int_0^T\int_\Omega \nabla.(H_\epsilon(\tilde{\theta}_\epsilon{}^n,\tilde{\varphi}_\epsilon{}^n)\nabla\tilde{\varphi}_\epsilon{}^n).\phi.$$

Therefore, the perturbed system has a solution $(\theta_\epsilon,\varphi_\epsilon)\in[L^\infty([0,T);L^\infty(\Omega))\cap L^2([0,T);H^1(\Omega))]^2$. By lemma 4.1, we know that there exists a constant $c=c(\zeta,a,\Omega,T,\theta_0)>0$ such that:

$$\inf_{[0,T)\times\Omega}\varphi_\epsilon\geq c=\zeta e^{-\lambda T}.$$

where c is independent of ϵ. So, we can pass to the limit in ϵ. Indeed, as previously, we may suppose that $(\theta_\epsilon,\varphi_\epsilon)\to(\theta,\varphi)$ weakly in $[L^\infty([0,T);L^\infty(\Omega))\cap L^2([0,T);H^1(\Omega))]^2$, strongly in $[L^\infty([0,T),L^q(\Omega))]^2$ with $(q<\infty)$, and a.e. in Q. The point wise convergence ensures that:

$$\theta\geq a>0 \quad \text{and} \quad \varphi\geq\zeta e^{-\lambda T}>0 \quad a.e.in \quad [0,T)\times\Omega, \tag{4.6}$$

because $\zeta e^{-\lambda T}$ does not depend on ϵ. The difficulty in this limit comes from the viscous term. We are saved by (4.6). Indeed, let x_1,x_2,y_1,y_2 be any functions such that:

$$x_1\geq a, x_2\geq a, y_1\geq c, y_2\geq c \quad a.e.inQ,$$

using the finite increments formula, we obtain:

$$\left|\frac{1}{(\epsilon^2+(x_1y_1)^2)^{1/2}}-\frac{1}{x_2y_2}\right|\leq C(a,c)\left|x_1y_1-x_2y_2\right|,$$

where $C(a,c)$ is a strictly positive constant which only depends on a and c and not on ϵ. Thus, we have existence in $L^\infty([0,T);L^\infty(\Omega))\cap L^2([0,T);H^1(\Omega))$.

□

Lemma 4.1

Under the hypothesis of theorem 1, $\tilde{\varphi}_\epsilon$ are positive and there exists a constant c independent of ϵ such that:

$$0<c(\zeta)=\zeta e^{-\lambda T}\leq\inf_{[0,T)\times\Omega}\varphi_\epsilon.$$

Proof of the lemma 4.1

The weak maximum principle is still valid for φ_ϵ by the fact that $\varphi_\epsilon^- = 0$ on $\partial\Omega$. Indeed, multiplying the φ_ϵ equation by $-\varphi_\epsilon^-$ and integrating, we obtain as in section 3 that $\varphi_\epsilon \geq 0$. On the other hand, it is obvious to see that $\varphi_\epsilon \leq b$.
The most important point is to prove the strong maximum principle which will be essential. Let,

$$\psi = \varphi_\epsilon - \zeta e^{-\lambda T},$$

$\lambda > 0$ will be specified in the following. The ψ equation is:

$$\frac{\partial\psi}{\partial t} + u\nabla\psi - \nabla.(H_\epsilon\nabla\psi) = -\psi A_\epsilon + \zeta(\lambda - A_\epsilon)e^{-\lambda T} \tag{4.7}$$

where

$$A_\epsilon = c_6\theta_\epsilon F + c_8 K_\epsilon(\theta_\epsilon) \geq 0.$$

The boundary and initial conditions are:

$$\psi_{\partial\Omega} = b - \zeta e^{-\lambda T}, \quad \psi(t=0) = \varphi_\epsilon(t=0) - \zeta.$$

We denote by ψ^- the negative part of ψ which vanishes on $\partial\Omega$. Multiplying (4.7) by $-\psi^-$ and integrating on Ω and due to the positivity of A_ϵ and H_ϵ, we have:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \psi^{-2} \leq -\int_\Omega \zeta(\lambda - A_\epsilon)e^{-\lambda T}\psi^-.$$

But, thanks to the estimates on θ_ϵ and the hypothesis on F, we have

$$A_\epsilon \in L^\infty([0,T); L^\infty(\Omega))$$

with

$$\|A_\epsilon\|_{L^\infty(\Omega)} \leq |\Omega|\,(a\|F\|_{L^\infty(\Omega)} + \frac{1}{a}). \tag{4.8}$$

Moreover, the bound does not depend on ϵ. So if we choose $\lambda \geq \|A_\epsilon\|_\infty$ using (4.8), we ensure that

$$\frac{d}{dt}\int_\Omega \psi^{-2} \leq 0.$$

But, since ψ^- is identically zero, we have clearly

$$\psi^- = 0 \quad a.e.inQ.$$

Therefore $\psi \geq 0a.e.inQ$ and

$$\varphi_\epsilon \geq \zeta e^{-\lambda T} > 0.$$

□

Lemma 4.2

If θ_0 and φ_0 are in $L^\infty(\Omega)$ then we have $(\theta_\epsilon, \varphi_\epsilon)$ in $[L^\infty([0,T]; L^\infty(\Omega)) \cap L^2([0,T]; H^1(\Omega))]^2$ and $(\theta'_\epsilon, \varphi'_\epsilon)$ in $[L^2([0,T]; H^{-1}(\Omega))]^2$.

Proof of Lemma 4.2

To avoid the treatment of boundary terms which appear in our analysis, we work in a translated system rather than the system (4.3)-(4.4). Indeed, introducing

$$\tilde{\theta}_\epsilon = \theta_\epsilon - a \quad \textit{and} \quad \tilde{\varphi}_\epsilon = \varphi_\epsilon - b,$$

we obtain:

$$\frac{\partial \tilde{\theta}_\epsilon}{\partial t} + u\nabla\tilde{\theta}_\epsilon - \nabla.(H_\epsilon(\tilde{\theta}_\epsilon + a, \tilde{\varphi}_\epsilon + b)\nabla\tilde{\theta}_\epsilon) = -c_3 F(\tilde{\theta}_\epsilon + a)^2 + c_5 \tag{4.9}$$

and

$$\frac{\partial \tilde{\varphi}_\epsilon}{\partial t} + u\nabla\tilde{\varphi}_\epsilon - \nabla.(H_\epsilon(\tilde{\theta}_\epsilon + a, \tilde{\varphi}_\epsilon + b)\nabla\tilde{\varphi}_\epsilon) = -(\tilde{\varphi}_\epsilon + b)(c_6(\tilde{\theta}_\epsilon + a)F + c_6 K_\epsilon(\tilde{\theta}_\epsilon + a)) \tag{4.10}$$

with

$$\tilde{\theta}(t=0) = \theta_0 - a, \quad \tilde{\varphi}(t=0) = \varphi_0 - b,$$
$$\tilde{\theta}_{\partial\Omega} = 0, \quad \tilde{\varphi}_{\partial\Omega} = 0$$

Thanks to the lemma 4.1, we notice that the viscous terms are bounded and that the bounds are not dependent of ϵ. Indeed, for all f in $H_0^1(\Omega)$ we have

$$-\int_\Omega \nabla.(H_\epsilon(\theta_\epsilon, \varphi_\epsilon)\nabla f).f \geq \mu \int_\Omega |\nabla f|^2 \tag{4.11}$$

and

$$-\int_\Omega \nabla.(H_\epsilon(\theta_\epsilon, \varphi_\epsilon)\nabla f).f \leq (\mu + \frac{e^{\lambda T}}{a\zeta}) \int_\Omega |\nabla f|^2. \tag{4.12}$$

So, applying the same method as in the constant eddy viscosity case, we deduce from (4.3) that:

$$\tilde{\theta}_\epsilon \in L^\infty([0,T]; L^\infty(\Omega)) \cap L^2([0,T]; H_0^1(\Omega)) \tag{4.13}$$

with bounds which depend only on θ_0. Concerning $\tilde{\varphi}_\epsilon$, we remark that:

$$0 < C(\theta_0) \leq K_\epsilon(\tilde{\theta}_\epsilon) \leq \frac{1}{a}.$$

So, the estimate (4.13) implies that

$$\tilde{\varphi}_\epsilon \in L^\infty([0,T]; L^\infty(\Omega)) \cap L^2([0,T]; H_0^1(\Omega)), \tag{4.14}$$

with bounds which depend only on φ_0 and θ_0.

Finally, $\tilde{\theta}_\epsilon{}'$ and $\tilde{\varphi}_\epsilon{}'$ are bounded in $L^2([0,T]; H^{-1}(\Omega))$.

$\square$

Remark

We are not able to prove that θ' and φ' are bounded in $L^2([0,T);L^2(\Omega))$, as in the case of constant eddy viscosity. Indeed, multiplying by something like $-D_{-h}D_h\theta$ may invoke terms of the form

$$\int_\Omega \frac{\varphi'\theta+\theta'\varphi}{(\theta\varphi)^2}\left|\nabla D_h\theta\right|^2,$$

on which the sign is not known a priori.

5 Numerical techniques

In this section, we give a brief description of the numerical techniques which we used. In particular, the numerical implementation of the $k-\varepsilon$ model is described. For more details the reader can refer to [MO1] and [MO2] where we have clearly shown that the $\varphi-\theta$ variables have a highly stabilizing behaviour.

We make a splitting between the Navier-Stokes and turbulent equations. We then use an iterative approach. This means that at each iteration, we use the mean values computed in the previous iteration to compute the new turbulent quantities. Then, we introduce these new values, specially the new eddy viscosity, in the Navier-Stokes solver.

To compute the turbulent quantities, we make a splitting between the diffusion and transport part of the $k-\varepsilon$ equations. For purposes of stability, we use dynamical parts of the $\varphi-\theta$ equations in the transport step rather the $k-varepsilon$ ones. Indeed, let k^n and ε^n be the values of k and ε at the previous iteration. We can compute θ^n and φ^n by

$$\theta^n=\frac{k^n}{\varepsilon^n}\quad \varphi^n=\frac{(\varepsilon^n)^2}{(k^n)^3}.$$

and then compute $\theta^{n+1/2}$ and $\varphi^{n+1/2}$ by (Δt being the time step):

$$\left(\frac{1}{\Delta t}+u^n\nabla.+\frac{(S_\theta^-)^n}{\theta^n}\right)\theta^{n+1/2}=\frac{\theta^n}{\Delta t}+(S_\theta^+)^n$$

and

$$\left(\frac{1}{\Delta t}+u^n\nabla.+\frac{(S_\varphi^-)^n}{\varphi^n}\right)\varphi^{n+1/2}=\frac{\varphi^n}{\Delta t}$$

where $(S_\theta^+)^n$, $(S_\theta^-)^n$ and $(S_\varphi^-)^n$ are the positive and negative parts of the right hand sides of the θ and φ equations ($S_\varphi^+=0$) evaluated explicitly. Once $\theta^{n+1/2}$ and $\varphi^{n+1/2}$ have been computed, we can return in $k-\varepsilon$ formulation by

$$k^{n+1/2}=\frac{1}{(\theta^{n+1/2})^2\varphi^{n+1/2}}\quad \varepsilon^{n+1/2}=\frac{1}{(\theta^{n+1/2})^3\varphi^{n+1/2}}.$$

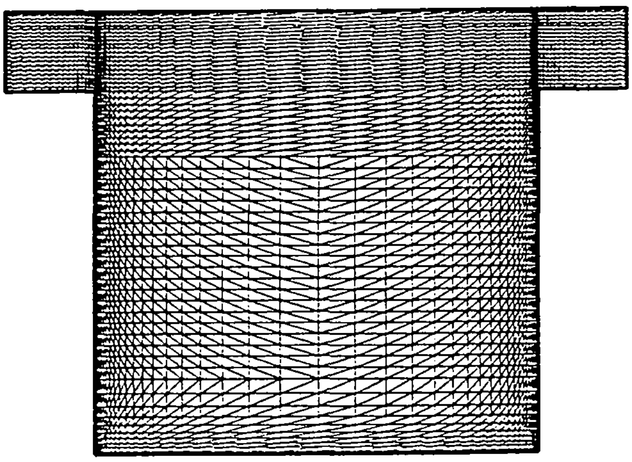
and take into account the viscous parts. The discretization in time is done using a characteristic scheme and in space using a classical P^1 conforming finite element approach. Mass lumping is used in the matrix to guarantee the positivity of the variables.

The low-Reynolds regions are treated by a two-layer approach as described in [MO2]. This algorithm is unconditionally stable and guarantees the positivity of the variables if there are no blunt angles in the mesh.

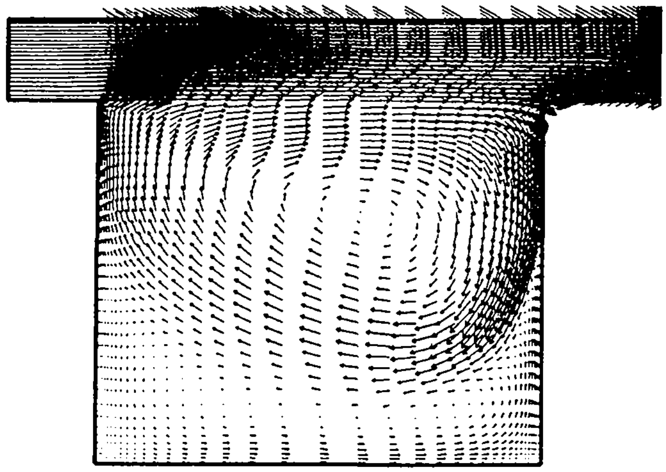
Flow in a cavity is presented as a test case. The Reynolds number based on the height of the cavity is $1.e^5$. At the inlet boundary, we take

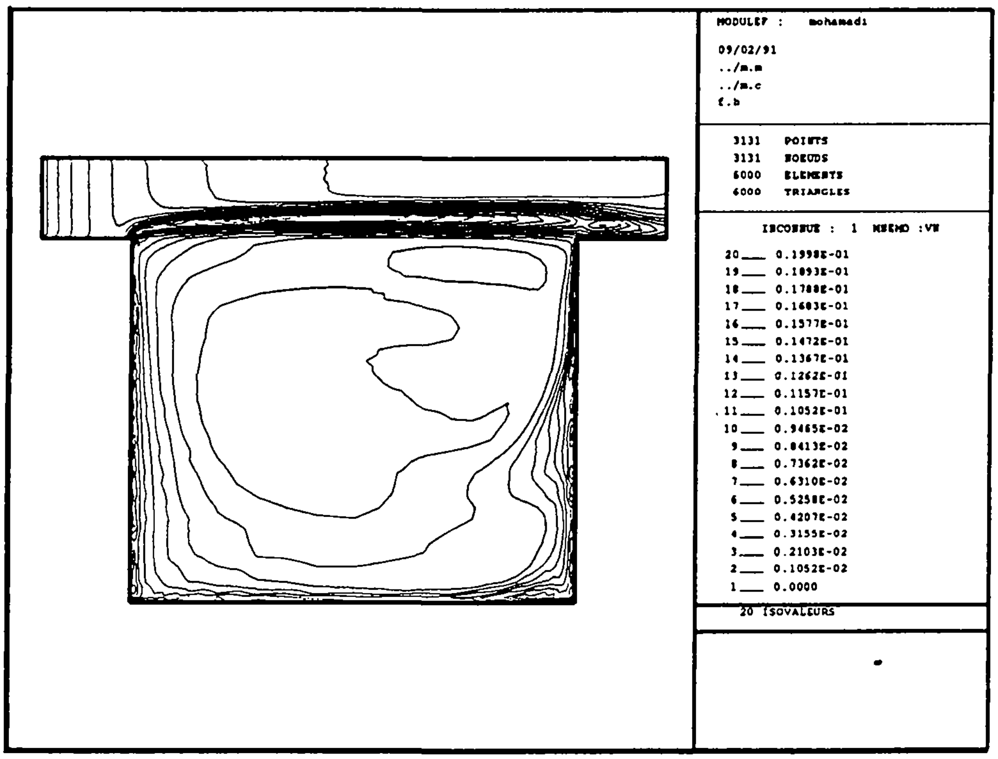
$$k_{in} = \varepsilon_{in} = 0.01 u_\infty^2$$

Moreover, the same computation is done with an inlet level of turbulence of $1.e^{-3} u_\infty^2$, but the results are qualitatively identical. As initial condition, we take a uniform state for all of the variables based on the inlet values. At the solid wall, u and k are frozen at zero. The Neumann boundary condition is taken at the outlet boundary. The figures show respectively the plots of u, k, ε and the eddy viscosity contours. The mesh has 3131 nodes and 6000 triangles.

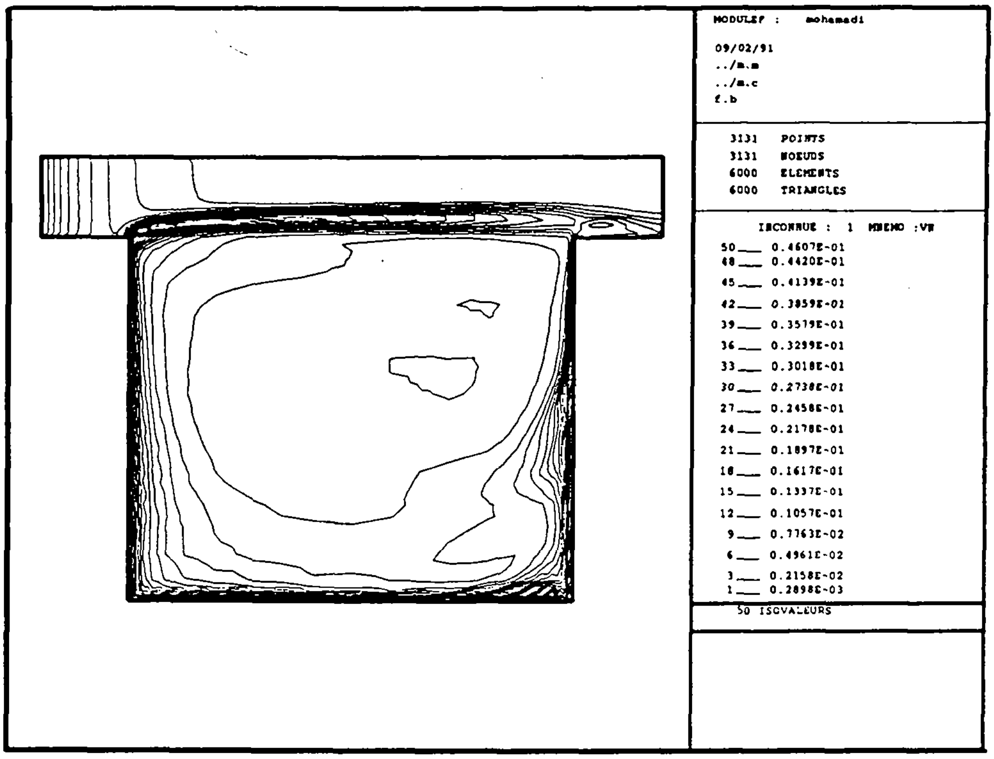


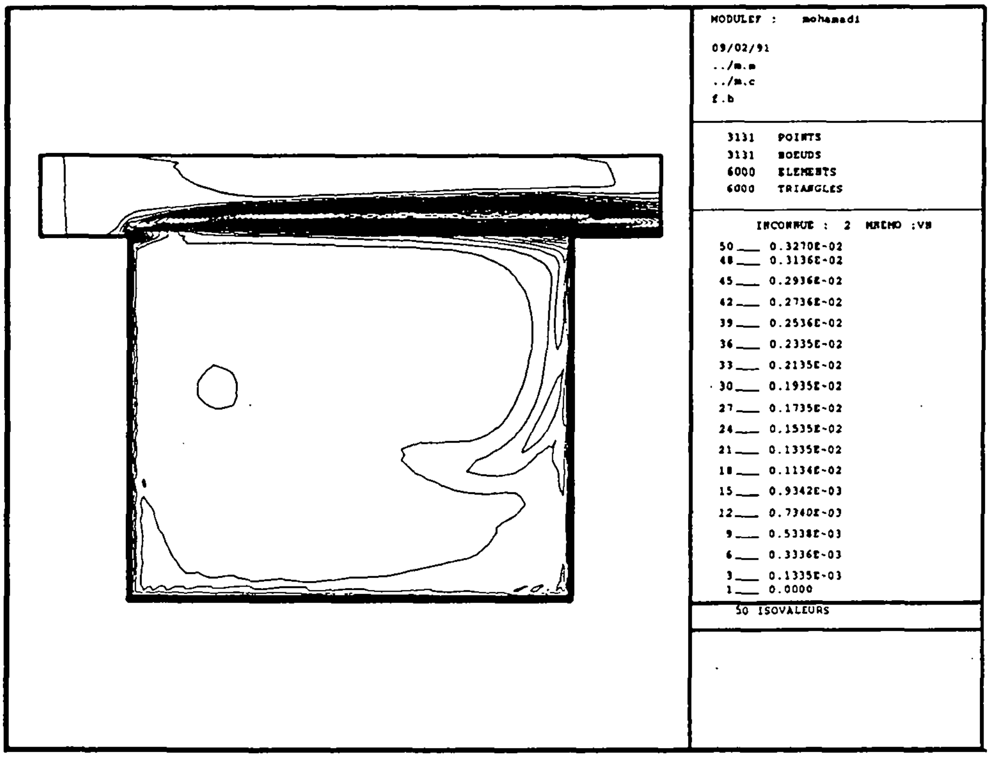
Velocity field (up) and turbulent kinetic energy k contours.





ε (up) and μ_t contours.





6 Conclusions

We have studied a two-equation $\varphi - \theta$ turbulence model from a mathematical point of view. Existence and positivity results are given for this model. Moreover, by the hypothesis of equivalence of the $\varphi-\theta$ and the $k-\varepsilon$ model, these existences and positivity results remain valid for the $k - \varepsilon$ model. In addition, it is shown that the $\varphi - \theta$ model is, mathematically speaking, clearly more avantageous than the $k - \varepsilon$ model. Indeed, in the $\varphi - \theta$ model, the right hand sides of the equations have the suitable sign.

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